

General (anti-)commutators of gamma matrices

Wolfgang Mück

*Dipartimento di Scienze Fisiche, Università degli Studi di Napoli “Federico II”
and INFN, Sezione di Napoli — via Cintia, 80126 Napoli, Italy*

E-mail: mueck@na.infn.it

Abstract

Commutators and anticommutators of gamma matrices with arbitrary numbers of (antisymmetrized) indices are derived.

Gamma matrix algebra is ubiquitous in many calculations in high energy physics. Whereas it is a fairly simple business in four dimensions, in higher-dimensional applications such as supergravity or M-theory, it becomes quite involved, because the number of independent matrices grows quickly with the number of space-time dimensions. To ease such calculations, on the one hand, one can resort to the help of computer algebra packages [1, and references therein]. On the other hand, one can look in the literature for reference tables, such as the appendix of [2], where the commutators and anti-commutators of gamma matrices with up to four indices are listed. The table in [2] is, to my knowledge, the most complete such list, but it contains typographical errors, as has been noticed in [3]. The purpose of this short note is to derive general formulae for the commutators and anti-commutators of gamma matrices with any number of indices. A general treatment is possible, because the (anti-)commutators do not depend on the space-time dimension, d , except for the fact that the number of indices any gamma matrix can carry is limited by d . The final formulae take the form of explicit sums and do not involve recursion relations.

Consider the d -dimensional Clifford algebra generated by the matrices γ^i ($i = 1, \dots, d$), which satisfy

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2g^{ij} , \quad (1)$$

where g^{ij} is the inverse metric tensor. Throughout this paper, indices shall be raised and lowered with g^{ij} and g_{ij} , respectively. Note that the metric g_{ij} can be curved or flat, and also its signature will be irrelevant for what follows. A useful basis of the Clifford algebra of matrices is given by the antisymmetrized products of the γ^i ,¹

$$\gamma^{i_1 \dots i_k} = \gamma^{[i_1} \gamma^{i_2} \dots \gamma^{i_k]} \quad (1 \leq k \leq d) , \quad (2)$$

¹The antisymmetrization includes a factor $1/k!$ for normalization. For odd d , the basis such defined is overcomplete, but this will not influence our analysis.

and by the identity matrix, which we may include in the notation (2) by allowing also for $k = 0$. We will formally allow also for $k > d$ implying that the corresponding matrix vanishes due to the antisymmetrization of the indices.

We shall obtain general formulae for all commutators and anti-commutators of the γ -matrices (2). A useful notation we will employ is the generalized commutator bracket

$$[a, b]_x = ab + xba , \quad (3)$$

where $x = \pm 1$. Our convention for the generalized Kronecker delta symbol is

$$\delta_{j_1 \dots j_k}^{i_1 \dots i_k} = \delta_{j_1}^{i_1} \dots \delta_{j_k}^{i_k} . \quad (4)$$

Let us start with the easiest piece and write

$$\begin{aligned} \gamma_j \gamma^{i_1 \dots i_k} &= \gamma_j \gamma^{[i_1 \dots i_k]} \\ &= -\gamma^{[i_1} \gamma_j \gamma^{i_2 \dots i_k]} + 2\delta_j^{[i_1} \gamma^{i_2 \dots i_k]} . \end{aligned}$$

After pulling γ_j through the other matrices, we end up with

$$[\gamma_j, \gamma^{i_1 \dots i_k}]_{(-1)^{k+1}} = 2k\delta_j^{[i_1} \gamma^{i_2 \dots i_k]} . \quad (5)$$

This is a commutator for even k and an anti-commutator for odd k . Finding the other bracket (anti-commutator for even k , commutator for odd k) is best done using induction. Let us assume that, for some k , the following relation holds:

$$[\gamma_j, \gamma^{i_1 \dots i_k}]_{(-1)^k} = 2\gamma_j^{i_1 \dots i_k} . \quad (6)$$

Consider $\gamma_j^{i_1 \dots i_{k+1}}$ and rewrite it as

$$2\gamma_j^{i_1 \dots i_{k+1}} = \frac{2}{k+2} \left(\gamma_j \gamma^{i_1 \dots i_{k+1}} - (k+1) \gamma^{[i_1} \gamma_j \gamma^{i_2 \dots i_{k+1}]} \right) . \quad (7)$$

Applying the hypothesis (6) on the second term in the parentheses and using then (1) and (5), one obtains after a bit of algebra

$$2\gamma_j^{i_1 \dots i_{k+1}} = \gamma_j \gamma^{i_1 \dots i_{k+1}} + (-1)^{k+1} \gamma^{i_1 \dots i_{k+1}} \gamma_j = [\gamma_j, \gamma^{i_1 \dots i_{k+1}}]_{(-1)^{k+1}} . \quad (8)$$

Thus, if the hypothesis (6) is valid for some k , then it will also hold for $k+1$. Therefore, as (6) holds for $k=1$ by the definition of γ^{ij} , we have shown that it holds for any k .

After this little exercise, we are ready to face the general cases $[\gamma_{j_1 \dots j_l}, \gamma^{i_1 \dots i_k}]_{\pm}$. Our hypotheses, which we shall prove again by induction, are the following:

$$[\gamma_{j_1 \dots j_l}, \gamma^{i_1 \dots i_k}]_{(-1)^{kl}} = 2 \sum_{m=0}^{\infty} (-1)^m (2m)! \binom{k}{2m} \binom{l}{2m} \delta_{[j_1 \dots j_{2m}}^{[i_1 \dots i_{2m}} \gamma_{j_{2m+1} \dots j_l]}^{i_{2m+1} \dots i_k]} , \quad (9)$$

$$\begin{aligned} [\gamma_{j_1 \dots j_l}, \gamma^{i_1 \dots i_k}]_{(-1)^{kl+1}} &= 2 \sum_{m=0}^{\infty} (-1)^{m+l+1} (2m+1)! \binom{k}{2m+1} \binom{l}{2m+1} \\ &\quad \times \delta_{[j_1 \dots j_{2m+1}}^{[i_1 \dots i_{2m+1}} \gamma_{j_{2m+2} \dots j_l]}^{i_{2m+2} \dots i_k]} . \end{aligned} \quad (10)$$

Notice that the sums in these formulae are actually not infinite, but they terminate, because the binomial coefficients vanish for large enough m . Similarly, we could have formally extended the sums to $-\infty$. It is straightforward to verify that (9) and (10) reduce to (6) and (5), respectively, if $l = 1$.

The proof by induction can be based on the identity

$$\begin{aligned} [\gamma_{j_1 \dots j_{l+1}}, \gamma^{i_1 \dots i_k}]_x &= \frac{1}{2} \left[\gamma_{j_{[1}}, \left[\gamma_{j_2 \dots j_{l+1}], \gamma^{i_1 \dots i_k} \right]_{x(-1)^k} \right]_{(-1)^{k+l}} \\ &+ \frac{1}{2} \left[\gamma_{j_{[1} \dots j_l}, \left[\gamma_{j_{l+1}], \gamma^{i_1 \dots i_k} \right]_{(-1)^{k+1}} \right]_{x(-1)^{k+l+1}}, \end{aligned} \quad (11)$$

which holds for $x^2 = 1$. Let us assume that (9) and (10) hold for some l and any k . Choosing $x = (-1)^{k(l+1)}$, we use (9) and (6) in the first term on the right hand side of (11), and (5) and (10) in the second term. Combining both terms (after shifting the summation index m by one in the second term), we end up with (9) with $(l+1)$ in place of l . Similarly, choosing $x = (-1)^{k(l+1)+1}$, we use (10) and (6) in the first term, (5) and (9) in the second term and obtain in the end (10) with $l+1$ in place of l . Therefore, as (9) and (10) hold for $l = 1$ and any k , we have shown that they hold for any k and l .

Equations (9) and (10) are the results of this paper. We leave it as an exercise for the reader to find three typographical errors in the list of (anti-)commutators given in the appendix of [2]. (In addition, a term containing a gamma matrix with eight indices has been omitted in the penultimate formula of this list.)

This research was supported by the European Commission, project MRTN-CT-2004-005104, and by the MiUR-COFIN project 2005-023102.

References

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